

## Section 11.10 Taylor and Maclaurin Series

Goal: find a power series representation for a general function  $f(x)$ . Let's

Let's assume that  $f(x) = C_0 + C_1(x-a) + C_2(x-a)^2 + C_3(x-a)^3 + \dots = \sum_{n=0}^{\infty} C_n(x-a)^n$

is the power series representation of  $f$ , with radius  $R$  ( $|x-a| < R$ ).

\* plugging  $x=a$  in  $f(x)$  yields  $f(a) = C_0$ . The derivative of  $f$  is  $f'(x) = 1 \cdot C_1 + 2C_2(x-a) + 3C_3(x-a)^2 + \dots = \sum_{n=1}^{\infty} nC_n(x-a)^{n-1}$ ,  $|x-a| < R$ .

\* plugging  $x=a$  in  $f'(x)$  yields  $f'(a) = 1 \cdot C_1$ . The 2<sup>nd</sup> derivative of  $f$  is  $f''(x) = 2 \cdot 1 \cdot C_2 + 3 \cdot 2 \cdot 1 (C_3(x-a) + \dots = \sum_{n=2}^{\infty} n(n-1)C_n(x-a)^{n-2}$ ,  $|x-a| < R$ .

\* plugging  $x=a$  in  $f''(x)$  yields  $f''(a) = 2 \cdot 1 \cdot C_2$ . The 3<sup>rd</sup> derivative of  $f$  is  $f'''(x) = 3 \cdot 2 \cdot 1 \cdot C_3 + 4 \cdot 3 \cdot 2 \cdot 1 C_4(x-a) + \dots = \sum_{n=3}^{\infty} n(n-1)(n-2)C_n(x-a)^{n-3}$ ,  $|x-a| < R$ .

\* plugging  $x=a$  in  $f'''(x)$  yields  $f'''(a) = 3 \cdot 2 \cdot 1 C_3$ .

In general, plugging  $x=a$  in the  $n^{\text{th}}$  derivative of  $f$ ,  $f^{(n)}$ , yields

$$f^{(n)}(a) = n(n-1)(n-2) \cdots 3 \cdot 2 \cdot 1 C_n, \text{ or } f^{(n)}(a) = n! C_n.$$

Thus the coefficients are given by  $C_n = \frac{f^{(n)}(a)}{n!}$

Theorem: If  $f$  has a power series representation at  $x=a$  given by

$$f(x) = \sum_{n=0}^{\infty} C_n(x-a)^n, |x-a| < R, \text{ then its coefficients are given by } C_n = \frac{f^{(n)}(a)}{n!}, n \geq 0.$$

\* So, the first few terms of the power series of  $f$  at  $x=a$  are

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \frac{f^{(4)}(a)}{4!}(x-a)^4 + \dots$$

This series is called the "Taylor Series of  $f$  at  $x=a$ " (or about  $x=a$ , or centered at  $x=a$ ).

Very often we take the center  $a=0$ ; The Taylor Series then becomes

$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3 + \dots$ . This Series is referred to as "the Maclaurin Series of  $f$ ".

Example ① Find the Maclaurin Series of  $f(x) = e^x$ , and its Radius of Conv.

Let  $f(x) = e^x$ ,  $f'(x) = e^x$ ,  $f''(x) = e^x$ , and in general,  $f^{(n)}(x) = e^x$ ,  $n \geq 0$ .

So  $f^{(n)}(0) = e^0 = 1$  for all  $n \geq 0 \Rightarrow c_n = \frac{f^{(n)}(0)}{n!} = \frac{1}{n!}$  for all  $n \geq 0$ .

The Maclaurin Series for  $e^x$  is therefore  $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \dots$

To find the radius of convergence, we use the ratio test. Let  $a_n = \frac{x^n}{n!}$ . Then

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = |x| \cdot \lim_{n \rightarrow \infty} \left| \frac{1}{n+1} \right| = 0 < 1 \text{ for all } x. \text{ Thus}$$

the Radius of Convergence is  $R = \infty$ , and the Maclaurin Series of  $e^x$  is

Valid for all real numbers  $x$ ! As a conclusion, we have

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \text{for all } x \quad \text{For } x=1, \text{ we get a formula for } e:$$

$$\begin{aligned} e = \sum_{n=0}^{\infty} \frac{1}{n!} &= 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \dots \\ &= 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \dots \end{aligned}$$

Let us now decompose the Taylor Series of  $f(x)$  as follows:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n = \sum_{n=0}^k \frac{f^{(n)}(a)}{n!} (x-a)^n + \sum_{n=k+1}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n.$$

The first sum is a partial sum (of the first  $k+1$  terms); it is a polynomial of degree  $K$ ; we call it  $T_k(x)$ .

$$\begin{aligned} \text{So, } T_k(x) &= \sum_{n=0}^k \frac{f^{(n)}(a)}{n!} (x-a)^n \\ &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(k)}(a)}{k!}(x-a)^k. \end{aligned}$$

$T_k(x)$  is called the  $K^{\text{th}}$  degree Taylor polynomial of " $f$ " at  $x=a$ .

for example, we have seen that  $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots$ . So

$$T_1(x) = 1+x, \quad T_2(x) = 1+x+\frac{x^2}{2!}, \quad T_3(x) = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}.$$

Back to the decomposition  $f(x) = T_k(x) + \sum_{n=k+1}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$ .

The sum on the right is the remainder  $R_k(x)$ . If we use  $T_k(x)$  as an approximation to  $f(x)$ , the error of the approx. is  $R_k(x)$ .

Below are two estimates of the remainder:

(I) Taylor's Inequality: if  $|f^{(k+1)}(x)| \leq M$  for  $|x-a| \leq d$ , then  $|R_k(x)| \leq \frac{M}{(k+1)!} |x-a|^{k+1}$  for  $|x-a| \leq d$ .

(II) Lagrange form of the remainder: there exists a number  $C$  between  $x$  and  $a$  such that  $R_k(x) = \frac{f^{(k+1)}(C)}{(k+1)!} (x-a)^{k+1}$ .

Example (2) What is the error made in estimating  $e^3$  using the 3rd degree Taylor polynomial of  $f(x) = e^x$  centered at (a)  $x=0$  (b)  $x=2$

$$(a) T_3(x) = \sum_{n=0}^{3!} \frac{f^{(n)}(0)}{n!} (x-0)^n = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} \Rightarrow e^3 \approx 1 + 3 + \frac{9}{2!} + \frac{27}{3!}.$$

$$R_3(x) = \frac{e^c}{(3+1)!} (x-0)^{3+1} \Rightarrow R_3(3) = \frac{e^c}{4!} \cdot 3^4, \text{ for } c \text{ in between } 3 \text{ and } 0; \text{ so } e^c \text{ is largest at } c=3 \Rightarrow R_3(3) \leq \frac{e^3}{4!} \cdot 3^4$$

$$(b) T_3(x) = \sum_{n=0}^{3!} \frac{f^{(n)}(2)}{n!} (x-2)^n = e^2 + \frac{e^2}{1!} (x-2) + \frac{e^2 (x-2)^2}{2!} + \frac{e^2 (x-2)^3}{3!}.$$

$$\text{So } e^3 \approx e^2 + e^2 + \frac{e^2}{2!} + \frac{e^2}{3!} \Rightarrow R_3(3) = \frac{e^c}{4!} \cdot (3-2)^4 = \frac{e^c}{4!}, \text{ for } c \text{ in between } 2 \text{ and } 3. \text{ so } e^c \text{ is largest at } c=3 \Rightarrow R_3(3) \leq \frac{e^3}{4!}$$

Examples ③ Find the MacLaurin Series of  $f(x) = \sin x$ .

$$f(x) = \sin x; f'(x) = \cos x; f''(x) = -\sin x; f'''(x) = -\cos x, f^{(4)}(x) = \sin x \dots$$

$$\Rightarrow f(0) = 0; f'(0) = 1; f''(0) = 0; f'''(0) = -1, f^{(4)}(0) = 0, \dots$$

In general,  $f^{(n)}(0) = \begin{cases} 0 & \text{if } n \text{ is even} \\ 1 \text{ or } -1 & \text{if } n \text{ is odd.} \end{cases}$  So,

$$\begin{aligned} \sin x &= f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3 + \frac{f^{(4)}(0)}{4!} x^4 + \dots \\ &= 0 + 1 \cdot \frac{x}{1!} + 0 \cdot \frac{x^2}{2!} + \frac{(-1)}{3!} x^3 + \frac{0}{4!} x^4 + \frac{1}{5!} x^5 + \dots \\ &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \dots \end{aligned}$$

Therefore,  $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$   $(-1)^n$  alternates the sign,  
radius is  $R = \infty$ .  $(2n+1)$  gives odd numbers only!

④ Find the MacLaurin Series of  $f(x) = \cos x$ .

Since  $\cos x = \frac{d}{dx} \sin x$ , we get  $\cos x = \frac{d}{dx} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ . So,

$$\begin{aligned} \cos x &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, R = \infty. \quad (\text{Even terms only!}) \\ &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} + \dots \end{aligned}$$

$$⑤ x^2 \sin x = x^2 \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+3}}{(2n+1)!} \quad \text{for all } x.$$

$$x \cos x = x \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+2}}{(2n+1)!} \quad \text{for all } x.$$

$$⑥ e^{-\frac{x^2}{2}} = \sum_{n=0}^{\infty} \left(-\frac{x^2}{2}\right)^n \cdot \frac{1}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^n \cdot n!} \quad \text{for all } x$$

⑦ Find the Maclaurin Series of  $f(x) = \cosh x$ .

$$\begin{aligned}
 f(x) &= \cosh(x) = \frac{1}{2} (e^x + e^{-x}) \\
 &= \frac{1}{2} \left( \sum_{n=0}^{\infty} \frac{x^n}{n!} + \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} \right) = \frac{1}{2} \left( \sum_{n=0}^{\infty} \frac{x^n}{n!} + \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!} \right) \\
 &= \frac{1}{2} \sum_{n=0}^{\infty} \frac{(1+(-1)^n)x^n}{n!}. \quad \text{Note: } 1+(-1)^n = \begin{cases} 2, & \text{if } n \text{ is even} \\ 0, & \text{if } n \text{ is odd} \end{cases} \\
 &= \frac{1}{2} \sum_{n=0}^{\infty} \frac{2 \cdot x^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}, \text{ for all } x.
 \end{aligned}$$

⑧ Find the 1<sup>st</sup> degree Taylor polynomial of  $f(x) = \tan^{-1}\left(\frac{x}{6}\right)$  at  $x=6$ .

$$f'(x) = \frac{1}{1 + \frac{x^2}{36}} \cdot \frac{1}{6}. \quad f(6) = \tan^{-1}(1) = \frac{\pi}{4}, \quad f'(6) = \frac{1}{12}. \quad \text{So,}$$

$$f(x) \approx T_1(x) = f(6) + f'(6)(x-6) = \frac{\pi}{4} + \frac{1}{12}(x-6).$$

This is the linear approximation of  $f(x)$  at  $x=6$ ; i.e.

$y = \frac{\pi}{4} + \frac{1}{12}(x-6)$  is the Eqn of the tangent line to  $f$  at  $x=6$ .

### The Binomial Series

Notation  $\binom{k}{n} = \frac{k \cdot (k-1) \cdot (k-2) \cdots (k-n+1)}{n!}$  is called the " $n^{\text{th}}$  binomial coefficient", and read as " $k$  choose  $n$ ";  $n$  is a positive integer.

$$\text{Examples } \binom{5}{2} = \frac{5(5-1)\cdots(5-2+1)}{2!} = \frac{5(5-1)}{2} = \frac{5 \cdot 4}{2} = 10$$

$$\binom{8}{3} = \frac{8(8-1)\cdots(8-3+1)}{3!} = \frac{8 \cdot 7 \cdot 6}{6} = 56$$

$$\binom{1/2}{4} = \frac{1/2(1/2-1)\cdots(1/2-4+1)}{4!} = \frac{1/2(-1/2)(-3/2)(-5/2)}{24}$$

$$\binom{-1/2}{3} = \frac{-1/2(-1/2-1)\cdots(-1/2-3+1)}{3!} = \frac{-1/2(-3/2)(-5/2)}{6}.$$

if  $k$  is any real number, and  $|x| < 1$ , the binomial Series of  $(1+x)^k$  is

$$\begin{aligned}
 (1+x)^k &= \sum_{n=0}^{\infty} \binom{k}{n} x^n \\
 &= \binom{k}{0} x^0 + \binom{k}{1} x^1 + \binom{k}{2} x^2 + \binom{k}{3} x^3 + \dots \\
 &= 1 + \frac{k}{1!} x + \frac{k(k-1)}{2!} x^2 + \frac{k(k-1)(k-2)}{3!} x^3 + \dots \\
 &= 1 + kx + \frac{k(k-1)}{2!} x^2 + \frac{k(k-1)(k-2)}{6} x^3 + \dots
 \end{aligned}$$

Examples ① find the first 4 terms of the binomial series for the function  $f(x) = \frac{1}{(1-\frac{x}{5})^5} = (1 + (-\frac{x}{5}))^{-5}$ . Here  $K = -5$ .

$$\begin{aligned}
 (1 + (-\frac{x}{5}))^{-5} &= \sum_{n=0}^{\infty} \binom{-5}{n} \left(-\frac{x}{5}\right)^n \\
 &= \binom{-5}{0} \left(-\frac{x}{5}\right)^0 + \binom{-5}{1} \left(-\frac{x}{5}\right)^1 + \binom{-5}{2} \left(-\frac{x}{5}\right)^2 + \binom{-5}{3} \left(-\frac{x}{5}\right)^3 + \dots \\
 &= 1 + \frac{-5}{1!} \left(-\frac{x}{5}\right) + \frac{-5(-5-1)}{2!} \left(-\frac{x}{5}\right)^2 + \frac{-5(-5-1)(-5-2)}{3!} \left(-\frac{x}{5}\right)^3 + \dots \\
 &= 1 + x + \frac{15 \cdot x^2}{25} + \frac{35 x^3}{125} + \dots
 \end{aligned}$$

the Series converges if  $|-\frac{x}{5}| < 1$ ; i.e. when  $|x| < 5$ . Radius is  $R=5$ .

② Find the first few terms of the MacLaurin Series of  $f(x) = \frac{2}{(1-x)^3}$ .

Method 1:  $\frac{2}{(1-x)^3} = \frac{d^2}{dx^2} \left(\frac{1}{1-x}\right) = \frac{d^2}{dx^2} (1+x+x^2+x^3+x^4+\dots) = 2+6x+12x^2+20x^3$

Method 2.  $\frac{2}{(1-x)^3} = 2(1-x)^{-3} = 2 \sum_{n=0}^{\infty} \binom{-3}{n} (-x)^n = \sum_{n=0}^{\infty} 2 \binom{-3}{n} (-1)^n x^n$

$$\begin{aligned}
 &= 2 \binom{-3}{0} (-1)^0 x^0 + 2 \binom{-3}{1} (-1)^1 x^1 + 2 \binom{-3}{2} (-1)^2 x^2 + 2 \binom{-3}{3} (-1)^3 x^3 + \dots \\
 &= 2 + 2(-3)(-x) + 2 \frac{-3(-3-1)}{2!} x^2 + 2 \frac{-3(-3-1)(-3-2)}{3!} (-x^3) + \dots \\
 &= 2 + 6x + 12x^2 + 20x^3 + \dots, \quad |x| < 1
 \end{aligned}$$

You get the same answer whether you use the geometric Series or the Binomial Series.

## Additional Examples

① Calculate  $\lim_{x \rightarrow 0} \frac{e^x - 1 - x}{x^2}$  without using l'Hopital's Rule.

Note that  $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots$  for all  $x$ .

Thus  $e^x - 1 - x = -1 - x + 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots = \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \dots$ ,

and so  $\frac{e^x - 1 - x}{x^2} = \frac{1}{x^2} \left( \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \dots \right) = \frac{1}{2} + \frac{x}{6} + \frac{x^2}{24} + \dots$  for all  $x$

then  $\lim_{x \rightarrow 0} \frac{e^x - 1 - x}{x} = \lim_{x \rightarrow 0} \left( \frac{1}{2} + \frac{x}{6} + \frac{x^2}{24} + \dots \right) = \frac{1}{2}$ .

② Find the first 3 nonzero terms of the Maclaurin Series for

$$F(x) = \int_0^x 2t^3 \cos t dt.$$

Method 1:  $F(x) = F(0) + F'(0)x + \frac{F''(0)}{2}x^2 + \dots$   
 $= 0 + 2 \cdot 0^3 \cos(0) \cdot x + \dots$ . this method is long!

Method 2:  $\cos t = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{(2n)!} = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \dots$

$$\text{So, } 2t^3 \cos t = 2t^3 \left( 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \dots \right) = 2t^3 - t^5 + \frac{t^7}{12} - \frac{t^9}{360} + \dots$$

$$\text{Thus } F(x) = \int_0^x \left( 2t^3 - t^5 + \frac{t^7}{12} - \frac{t^9}{360} + \dots \right) dt = \left( \frac{2t^4}{4} - \frac{t^6}{6} + \frac{t^8}{96} - \dots \right) \Big|_0^x$$

$$= \frac{x^4}{2} - \frac{x^6}{6} + \frac{x^8}{96} - \dots$$